

The a priori $\tan \Theta$ theorem for spectral subspaces*

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Abstract. Let A be a self-adjoint operator on a separable Hilbert space \mathfrak{H} . Assume that the spectrum of A consists of two disjoint components σ_0 and σ_1 such that the set σ_0 lies in a finite gap of the set σ_1 . Let V be a bounded self-adjoint operator on \mathfrak{H} off-diagonal with respect to the partition $\text{spec}(A) = \sigma_0 \cup \sigma_1$. It is known that if $\|V\| < \sqrt{2}d$, where $d = \text{dist}(\sigma_0, \sigma_1)$, then the perturbation V does not close the gaps between σ_0 and σ_1 and the spectrum of the perturbed operator $L = A + V$ consists of two isolated components ω_0 and ω_1 originating from σ_0 and σ_1 , respectively. Furthermore, it is known that if V satisfies the stronger bound $\|V\| < d$ then for the difference of the spectral projections $E_A(\sigma_0)$ and $E_L(\omega_0)$ of A and L associated with the spectral sets σ_0 and ω_0 , respectively, the following sharp norm estimate holds:

$$\|E_A(\sigma_0) - E_L(\omega_0)\| \leq \sin \left(\arctan \frac{\|V\|}{d} \right).$$

In the present work we prove that this estimate remains valid and sharp also for $d \leq \|V\| < \sqrt{2}d$, which completely settles the issue.

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1. Introduction

An important problem in the perturbation theory of self-adjoint operators is to study the variation of the spectral subspace associated with an isolated spectral subset that is subject to a perturbation (see, e.g., [7]). Classical trigonometric estimates in subspace perturbation problem have been established by Davis and Kahan [5]. For further results on subspace variation bounds for self-adjoint operators we refer to [2], [6], [11], [12], [13] and the references therein.

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In this article we consider a self-adjoint operator A on a separable Hilbert space \mathfrak{H} , assuming that the spectrum of A consists of two disjoint components σ_0 and σ_1 such that the set σ_0 lies in a finite gap of the set σ_1 . In other words, we suppose that

$$\overline{\text{conv}(\sigma_0)} \cap \overline{\sigma_1} = \emptyset \quad \text{and} \quad \sigma_0 \subset \text{conv}(\sigma_1), \quad (1.1)$$

where conv denotes the convex hull and overlining means closure. The perturbations V are assumed to be bounded and off-diagonal with respect to the partition $\text{spec}(A) = \sigma_0 \cup \sigma_1$, that is, V should anticommute with the difference $E_A(\sigma_0) - E_A(\sigma_1)$ of the spectral projections $E_A(\sigma_0)$ and $E_A(\sigma_1)$ of A associated with the sets σ_0 and σ_1 , respectively. For the spectral disposition (1.1), it has been proven in [9] (see also [15, 16]) that the gaps between σ_0 and σ_1 remain open if the off-diagonal self-adjoint perturbation V satisfies the (sharp) condition

$$\|V\| < \sqrt{2} d, \quad (1.2)$$

where $d := \text{dist}(\sigma_0, \sigma_1)$ stands for the distance between σ_0 and σ_1 . Under this condition the spectrum of the perturbed operator $L = A + V$ consists of two isolated components $\omega_0 \subset \Delta$ and $\omega_1 \subset \mathbb{R} \setminus \Delta$. Here and in the sequel, Δ denotes the finite gap of σ_1 that contains σ_0 . (We recall that by a finite gap of a closed set $\sigma \subset \mathbb{R}$ one understands an open bounded interval on \mathbb{R} that does not intersect this set but both ends of which belong to σ .) It is worth noting that the norm bound (1.2) is also optimal in the sense that, if it is violated, the spectrum of L in the gap Δ may be completely empty (see [10, Example 1.6]).

The goal of the present paper consists in finalizing a sharp norm estimate on the variation of the spectral subspace $\text{Ran}(E_A(\sigma_0))$ under off-diagonal self-adjoint perturbations that was conjectured and partly proven in [13]. Our main result is as follows.

Theorem 1. *Given a (possibly unbounded) self-adjoint operator A on a separable Hilbert space \mathfrak{H} , assume that its spectrum consists of two disjoint components σ_0 and σ_1 satisfying condition (1.1). Let V be a bounded self-adjoint operator on \mathfrak{H} off-diagonal with respect to the partition $\text{spec}(A) = \sigma_0 \cup \sigma_1$ and set $L = A + V$, $\text{Dom}(L) = \text{Dom}(A)$. Assume in addition that V satisfies the bound (1.2) and let $\omega_0 = \text{spec}(L) \cap \Delta$. Then the difference between the spectral projections $E_A(\sigma_0)$ and $E_L(\omega_0)$ of A and L associated with the respective spectral sets σ_0 and ω_0 satisfies the norm estimate*

$$\|E_A(\sigma_0) - E_L(\omega_0)\| \leq \sin \left(\arctan \frac{\|V\|}{d} \right) \quad \left(< \sqrt{\frac{2}{3}} \right). \quad (1.3)$$

We underline that for $\|V\| < d$ the bound (1.3) was established in [13]. It was called there the *A priori tan Θ Theorem*. For $\|V\| = d$ this bound may be obtained from the result of [13] by continuity. Having proved Theorem 1 we confirm the truth of the conjecture of [13, Remark 5.7] and we thus close the gap in the subspace perturbation problem for dispositions (1.1) which has remained for $\|V\|/d \in (1, \sqrt{2})$. We also remark that the a priori tan θ theorem for eigenvectors [3, Theorem 1.1] is a simple corollary of Theorem 1.

Our proof of Theorem 1 is essentially based on the reduction of the subspace perturbation problem under consideration to the study of the operator Riccati equation

$$XA_0 - A_1X + XBX = B^* \quad (1.4)$$

with $A_0 = A|_{\mathfrak{A}_0}$, $A_1 = A|_{\mathfrak{A}_1}$, and $B = V|_{\mathfrak{A}_1}$ where $\mathfrak{A}_0 = \text{Ran}(E_A(\sigma_0))$ and $\mathfrak{A}_1 = \text{Ran}(E_A(\sigma_1))$. In fact, the perturbed spectral subspace $\mathfrak{L}_0 = \text{Ran}(E_L(\omega_0))$ is the graph of a particular solution $X \in \mathcal{B}(\mathfrak{A}_0, \mathfrak{A}_1)$ to equation (1.4). In such a case (see, e.g., [8])

$$\|E_A(\sigma_0) - E_L(\omega_0)\| = \sin(\arctan \|X\|). \quad (1.5)$$

Thus, having established a bound for the solution X one simultaneously obtains an estimate for the norm of the difference of the spectral projections $E_A(\sigma_0)$ and $E_L(\omega_0)$ as well as a bound for the operator angle

$$\Theta = \arctan \sqrt{X^*X} \quad (1.6)$$

between the spectral subspaces \mathfrak{A}_0 and \mathfrak{L}_0 . For the concept of operator angle and related material we refer to [8] and references therein. Note that because of (1.6) the operator X itself is usually called the angular operator for the pair of subspaces $(\mathfrak{A}_0, \mathfrak{L}_0)$.

By (1.5) and (1.6), the bound (1.3) can be equivalently written in the form

$$\tan \Theta \leq \frac{\|V\|}{d}$$

which implies that under conditions (1.1) and (1.2) the norm of the operator angle between \mathfrak{A}_0 and \mathfrak{L}_0 can never exceed the value of $\arctan \sqrt{2}$ ($\approx 54^\circ 44'$).

The present article is the third in a series of papers on a priori $\tan \Theta$ bounds, following [3, 13]. Its strategy, however, is very different from the approaches used in [3] and [13]. The approach of paper [13] (which was actually the first in the series) is based on the properties of sectorial operators and on an involution technique that works only in cases where $\Theta < \pi/4$ (also cf. [6]) and the corresponding angular operators X in (1.5) are contractions. The approach of [3] only applies to individual eigenvectors of L and there is no chance to extend it to multi-dimensional spectral subspaces. The key ingredient of the method we use in this paper is a new identity for eigenvalues and eigenvectors of the modulus $|X| = \sqrt{X^*X}$ of X that was found only after the articles [3] and [13] were written. Here we mean the identity (2.9) of Lemma 2.2 below which allows us to obtain a norm bound for X even if X is not a contraction (see Theorem 3.2 and its proof).

The paper is organized as follows. In Section 2 we prove the key Lemma 2.2. Then we recall some known bounds on the shift of the spectrum of the operator A under a perturbation V satisfying the more detailed (and weaker than (1.2)) condition $\|V\| < \sqrt{d|\Delta|}$ where $|\Delta|$ stands for the length of the gap Δ . We also recall a known norm bound for the angular operator X in (1.5) that is valid for $\|V\| < \sqrt{d(|\Delta| - d)}$. In Section 3 we employ the identity (2.9) to obtain an estimate for $\|X\|$ already for $\|V\| \geq \sqrt{d(|\Delta| - d)}$ but in the special case where $|X|$ is assumed to have an eigenvalue equal to $\|X\|$. In Section 4, this estimate for $\|X\|$ is used to prove our most

general and detailed subspace variation bound (see Theorem 4.1). We conclude with a proof of Theorem 1.

The following notations are used throughout the paper. By a subspace we always understand a closed linear subset of a Hilbert space. The identity operator on a subspace (or on the whole Hilbert space) \mathfrak{M} is denoted by $I_{\mathfrak{M}}$. If no confusion arises, the index \mathfrak{M} may be omitted in this notation. The Banach space of bounded linear operators from a Hilbert space \mathfrak{M} to a Hilbert space \mathfrak{N} is denoted by $\mathcal{B}(\mathfrak{M}, \mathfrak{N})$. By $\mathfrak{M} \oplus \mathfrak{N}$ we understand the orthogonal sum of two Hilbert spaces (or orthogonal subspaces) \mathfrak{M} and \mathfrak{N} . The graph $\mathcal{G}(K) = \{y \in \mathfrak{M} \oplus \mathfrak{N} \mid y = x \oplus Kx, x \in \mathfrak{M}\}$ of a bounded operator $K \in \mathcal{B}(\mathfrak{M}, \mathfrak{N})$ is called the graph subspace (associated with the operator K). By $E_T(\sigma)$ we always denote the spectral projection of a self-adjoint operator T associated with a Borel set $\sigma \subset \mathbb{R}$. The notation $\rho(T)$ is used for the resolvent set of T . The domain and the range of an operator S are denoted by $\text{Dom}(S)$ and $\text{Ran}(S)$, respectively.

2. Preliminaries

It is convenient to represent the operators under consideration as block operator matrices. Since condition (1.1) will not always be assumed, we first adopt a hypothesis that implies no constraints on the mutual position of the spectra of the entries A_0 and A_1 .

Hypothesis 2.1. *Let \mathfrak{A}_0 and \mathfrak{A}_1 be complementary orthogonal subspaces of a separable Hilbert space \mathfrak{H} . Assume that A is a self-adjoint operator on $\mathfrak{H} = \mathfrak{A}_0 \oplus \mathfrak{A}_1$ admitting the block diagonal representation*

$$A = \begin{pmatrix} A_0 & 0 \\ 0 & A_1 \end{pmatrix}, \quad \text{Dom}(A) = \mathfrak{A}_0 \oplus \text{Dom}(A_1), \quad (2.1)$$

with A_0 a bounded self-adjoint operator on \mathfrak{A}_0 and A_1 a possibly unbounded self-adjoint operator on \mathfrak{A}_1 . Suppose that V is an off-diagonal bounded self-adjoint operator on \mathfrak{H} , i.e.,

$$V = \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix}, \quad (2.2)$$

where $0 \neq B \in \mathcal{B}(\mathfrak{A}_1, \mathfrak{A}_0)$, and let $L = A + V$, $\text{Dom}(L) = \text{Dom}(A)$, that is,

$$L = \begin{pmatrix} A_0 & B \\ B^* & A_1 \end{pmatrix}, \quad \text{Dom}(L) = \mathfrak{A}_0 \oplus \text{Dom}(A_1). \quad (2.3)$$

Under the assumptions of Hypothesis 2.1, an operator $X \in \mathcal{B}(\mathfrak{A}_0, \mathfrak{A}_1)$ is said to be a solution of the operator Riccati equation (1.4) if

$$\text{Ran}(X) \subset \text{Dom}(A_1) \quad (2.4)$$

and (1.4) holds as an operator equality on \mathfrak{A}_0 (cf., e.g., [1, Definition 3.1]). Clearly, the solution X , whenever it exists, satisfies $X \neq 0$; otherwise, $X = 0$ implies $B = 0$ which contradicts the hypothesis. In the following by U we denote the partial

isometry in the polar decomposition $X = U|X|$ of X . We adopt the convention that U is extended to $\text{Ker}(X) = \text{Ker}(|X|)$ by

$$U|_{\text{Ker}(X)} = 0. \quad (2.5)$$

In such a case U is uniquely defined on the whole space \mathfrak{A}_0 (see, e.g., [4, Theorem 8.1.2]) and

$$U \text{ is an isometry on } \text{Ran}(|X|) = \text{Ran}(X^*). \quad (2.6)$$

The assertion below provides us with three useful identities for eigenvalues and eigenvectors (in case they exist) of the modulus $|X|$.

Lemma 2.2. *Assume Hypothesis 2.1. Let $X \in \mathcal{B}(\mathfrak{A}_0, \mathfrak{A}_1)$ be a solution to the operator Riccati equation (1.4). Suppose that $|X|$ has an eigenvalue λ ($\lambda \geq 0$) and that u , $u \neq 0$, is an eigenvector of $|X|$ corresponding to this eigenvalue, i.e. $|X|u = \lambda u$. If U is the isometry from the polar representation $X = U|X|$ of the operator X , then $Uu \in \text{Dom}(A_1)$ and the following three identities hold:*

$$\begin{aligned} \lambda (\|A_0 u\|^2 + \|B^* u\|^2 - \|A_1 U u\|^2 - \|B U u\|^2) \\ = (1 - \lambda^2) (\langle A_0 u, B U u \rangle + \langle B^* u, A_1 U u \rangle), \end{aligned} \quad (2.7)$$

$$\lambda (\langle A_0 u, B U u \rangle + \langle B^* u, A_1 U u \rangle) = \|\Lambda_0 u\|^2 - \|A_0 u\|^2 - \|B^* u\|^2, \quad (2.8)$$

$$\lambda^2 (\|A_1 U u\|^2 + \|B U u\|^2 - \|\Lambda_0 u\|^2) = \|A_0 u\|^2 + \|B^* u\|^2 - \|\Lambda_0 u\|^2, \quad (2.9)$$

where the entry

$$\Lambda_0 = (I + |X|^2)^{1/2} (A_0 + B X) (I + |X|^2)^{-1/2} \quad (2.10)$$

is bounded and self-adjoint on \mathfrak{A}_0 .

Proof. We start with remark that if $\lambda \neq 0$ then $Uu = \frac{1}{\lambda} U|X|u = \frac{1}{\lambda} Xu$ and, hence, $Uu \in \text{Dom}(A_1)$ by (2.4). For $\lambda = 0$ we have $u \in \text{Ker}(|X|) = \text{Ker}(X)$ and then $Uu = 0 \in \text{Dom}(A_1)$ by convention (2.5). We also notice that for the eigenvector u of $|X|$ associated with the nonzero eigenvalue $\lambda > 0$ one automatically has $u \in \text{Ran}(|X|)$ and, thus, in this case the assertion (2.6) implies $U^* U u = u$.

First we prove the identity (2.7). If $\lambda = 0$, then (2.7) is trivial since $Uu = 0$ due to (2.5). Suppose that $\lambda > 0$ and set

$$x := \begin{pmatrix} u \\ Xu \end{pmatrix} = \begin{pmatrix} u \\ U|X|u \end{pmatrix} = \begin{pmatrix} u \\ \lambda Uu \end{pmatrix}, \quad (2.11)$$

$$y := \begin{pmatrix} -X^* U u \\ U u \end{pmatrix} = \begin{pmatrix} -|X| U^* U u \\ U u \end{pmatrix} = \begin{pmatrix} -|X| u \\ U u \end{pmatrix} = \begin{pmatrix} -\lambda u \\ U u \end{pmatrix}. \quad (2.12)$$

From $Uu \in \text{Dom}(A_1)$ one concludes that both x and y belong to $\text{Dom}(L)$. Since X is a solution to the operator Riccati equation (1.4), by, e.g., [1, Lemma 5.3 and Theorem 5.5] the graphs $\mathcal{G}(X)$ and $\mathcal{G}(-X^*)$ are reducing subspaces for the operator matrix L . Clearly, $x \in \mathcal{G}(X)$ and $y \in \mathcal{G}(-X^*)$ which yields $Lx \in \mathcal{G}(X)$ and $Ly \in \mathcal{G}(-X^*)$. Since the subspaces $\mathcal{G}(X)$ and $\mathcal{G}(-X^*)$ are orthogonal to each other, we have

$$\langle Lx, Ly \rangle = 0. \quad (2.13)$$

Using the last equalities in (2.11) and (2.12) one obtains

$$Lx = \begin{pmatrix} A_0u + \lambda BUu \\ B^*u + \lambda A_1Uu \end{pmatrix} \quad \text{and} \quad Ly = \begin{pmatrix} -\lambda A_0u + BUu \\ -\lambda B^*u + A_1Uu \end{pmatrix}. \quad (2.14)$$

Substitution of the expressions for Lx and Ly from (2.14) into the equality (2.13) results in the identity (2.7).

To prove (2.8), we begin with the following equalities:

$$A_0BX + BA_1X = A_0BX + B(XA_0 + XBX - B^*) \quad (2.15)$$

$$= (A_0 + BX)^2 - A_0^2 - BB^*, \quad (2.16)$$

by taking into account at the step (2.15) that, due to (1.4), $A_1X = XA_0 + XBX - B^*$. Since $Xu = U|X|u = \lambda Uu$ and $Uu \in \text{Dom}(A_1)$, equality (2.16) yields

$$\lambda (\langle A_0u, BUu \rangle + \langle B^*u, A_1Uu \rangle) = \langle u, (A_0 + BX)^2u \rangle - \|A_0u\|^2 - \|B^*u\|^2. \quad (2.17)$$

Clearly,

$$(A_0 + BX)^2 = (I + |X|^2)^{-1/2} \Lambda_0^2 (I + |X|^2)^{1/2}, \quad (2.18)$$

where Λ_0 is the bounded operator given by (2.10). Since u is an eigenvector of $|X|$, by (2.18) one obtains

$$\begin{aligned} \langle u, (A_0 + BX)^2u \rangle &= \langle (I + |X|^2)^{-1/2}u, \Lambda_0^2 (I + |X|^2)^{1/2}u \rangle \\ &= \langle (1 + \lambda^2)^{-1/2}u, \Lambda_0^2 (1 + \lambda^2)^{1/2}u \rangle \\ &= \langle u, \Lambda_0^2u \rangle. \end{aligned} \quad (2.19)$$

That the operator Λ_0 is self-adjoint follows, e.g., from [1, Theorem 5.5]. Hence, combining (2.17) and (2.19) one arrives at (2.8).

As for the identity (2.9), for $\lambda = 0$ it follows immediately from (2.8). If $\lambda > 0$, then (2.9) is obtained by combining (2.8) with (2.7). \square

From now on we assume the spectral disposition (1.1). When necessary, this disposition will be described in more detail as follows.

Hypothesis 2.3. Assume Hypothesis 2.1. Let $\sigma_0 = \text{spec}(A_0)$ and $\sigma_1 = \text{spec}(A_1)$. Suppose that an open interval $\Delta = (\gamma_l, \gamma_r) \subset \mathbb{R}$, $\gamma_l < \gamma_r$, is a finite gap of the set σ_1 and $\sigma_0 \subset \Delta$. Set $d = \text{dist}(\sigma_0, \sigma_1)$.

Below we will use the following assertions obtained by using several results proven in [9].

Theorem 2.4. Assume Hypothesis 2.3 and suppose that $\|V\| < \sqrt{d|\Delta|}$. Then:

- (i) The spectrum of the block operator matrix L consists of two disjoint components $\omega_0 \subset \Delta$ and $\omega_1 \subset \mathbb{R} \setminus \Delta$. In particular,

$$\min(\omega_0) \geq \gamma_l + (d - r_V) \quad \text{and} \quad \max(\omega_0) \leq \gamma_r - (d - r_V), \quad (2.20)$$

where

$$r_V := \|V\| \tan \left(\frac{1}{2} \arctan \frac{2\|V\|}{|\Delta| - d} \right) < d. \quad (2.21)$$

- (ii) *There is a unique solution $X \in \mathcal{B}(\mathfrak{A}_0, \mathfrak{A}_1)$ to the Riccati equation (1.4) with the properties*

$$\operatorname{spec}(A_0 + BX) = \omega_0 \text{ and } \operatorname{spec}(A_1 - B^*X^*) = \omega_1; \quad (2.22)$$

the spectral subspaces $\mathfrak{L}_0 = \operatorname{Ran}(E_L(\omega_0))$ and $\mathfrak{L}_1 = \operatorname{Ran}(E_L(\omega_1))$ are graph subspaces of the form $\mathfrak{L}_0 = \mathcal{G}(X)$ and $\mathfrak{L}_1 = \mathcal{G}(-X^)$.*

Remark 2.5. Assertion (i) of Theorem 2.4 follows from [9, Theorem 3.2]. Assertion (ii) is obtained by combining [9, Theorem 2.3] with an existence and uniqueness result for the operator Riccati equation (1.4) established in [9, Theorem 1 (i)].

A sharp a priori norm estimate for the operator angle between the subspaces $\operatorname{Ran}(E_A(\sigma_0))$ and $\operatorname{Ran}(E_L(\omega_0))$ and, equivalently, for the corresponding angular operator X in (1.5) was obtained in [13, Theorem 5.3] under an assumption that is stronger than condition $\|V\| < \sqrt{d|\Delta|}$ of Theorem 2.4. We formulate the main statement of [13, Theorem 5.3] in the following form.

Theorem 2.6 ([13]). *Assume Hypothesis 2.3. Assume in addition that*

$$\|V\| < \sqrt{d(|\Delta| - d)}.$$

Let X be the unique solution to the Riccati equation (1.4) with the properties (2.22). Then

$$\|X\| \leq \tan \left(\frac{1}{2} \arctan \kappa(|\Delta|, d, \|V\|) \right) \quad (< 1), \quad (2.23)$$

where $\kappa(D, d, v)$ is defined for

$$D > 0, \quad 0 < d \leq \frac{D}{2}, \quad \text{and} \quad 0 \leq v < \sqrt{d(D-d)} \quad (2.24)$$

by

$$\kappa(D, d, v) := \begin{cases} \frac{2v}{d} & \text{if } v \leq \frac{1}{2} \sqrt{d(D-2d)}, \\ \frac{vD + \sqrt{d(D-d)} \sqrt{(D-2d)^2 + 4v^2}}{2(d(D-d) - v^2)} & \text{if } v > \frac{1}{2} \sqrt{d(D-2d)}. \end{cases}$$

In the sequel, the estimating function appearing on the right-hand side of (2.23) will be denoted by M_1 , that is,

$$M_1(D, d, v) := \tan \left(\frac{1}{2} \arctan \kappa(D, d, v) \right), \quad (D, d, v) \in \Omega_1, \quad (2.25)$$

where Ω_1 stands for the set of points $(D, d, v) \in \mathbb{R}^3$ with coordinates D, d, v satisfying (2.24).

Remark 2.7. Using the elementary formula

$$\tan \left(\frac{1}{2} \arctan x \right) = \frac{x}{1 + \sqrt{1 + x^2}}, \quad x \in \mathbb{R},$$

one can also write the function $M_1(D, d, v)$ in the algebraic form

$$M_1(D, d, v) \Big|_{\Omega_1^{(0)}} = \frac{2v}{d + \sqrt{d^2 + 4v^2}}, \quad (2.26)$$

$$M_1(D, d, v) \Big|_{\Omega_1^{(1)}} = \frac{v(2v + \sqrt{(D-2d)^2 + 4v^2}) + \sqrt{d(D-d)}(D - 2\sqrt{d(D-d)})}{Dv + \sqrt{d(D-d)}\sqrt{(D-2d)^2 + 4v^2}}, \quad (2.27)$$

where $\Omega_1^{(0)}$ and $\Omega_1^{(1)}$ denote the corresponding complementary parts of the set Ω_1 ,

$$\begin{aligned} \Omega_1^{(0)} &:= \left\{ (D, d, v) \in \Omega_1 \mid 0 \leq v \leq \frac{1}{2}\sqrt{d(D-2d)} \right\}, \\ \Omega_1^{(1)} &:= \left\{ (D, d, v) \in \Omega_1 \mid \frac{1}{2}\sqrt{d(D-2d)} < v < \sqrt{d(D-d)} \right\}. \end{aligned}$$

By (2.25) we have

$$0 \leq M_1(D, d, v) < 1 \quad \text{for any } (D, d, v) \in \Omega_1.$$

By representation (2.27) the function $M_1(D, d, v)$ admits a continuous extension to the part

$$\partial\Omega_{12} := \left\{ (D, d, v) \in \mathbb{R}^3 \mid D > 0, 0 < d \leq D/2, v = \sqrt{d(D-d)} \right\} \quad (2.28)$$

of the boundary of Ω_1 where $v = \sqrt{d(D-d)}$. For the extended function we keep the same notation M_1 . One verifies by inspection that $M_1(D, d, v) = 1$ for any $(D, d, v) \in \partial\Omega_{12}$.

Obviously, the function $M_1(D, d, v)$ is infinitely differentiable within the sets $\Omega_1^{(0)}$ and $\Omega_1^{(1)}$. Furthermore, this function and the partial derivatives $\frac{\partial M_1(D, d, v)}{\partial D}$, $\frac{\partial M_1(D, d, v)}{\partial d}$, and $\frac{\partial M_1(D, d, v)}{\partial v}$ vary continuously when (D, d, v) passes through the common border $\partial\Omega_1^{(01)} = \Omega_1^{(0)} \cap \Omega_1^{(1)}$ of the subsets $\Omega_1^{(0)}$ and $\Omega_1^{(1)}$. Thus, the function M_1 and its derivatives $\frac{\partial M_1}{\partial D}$, $\frac{\partial M_1}{\partial d}$, and $\frac{\partial M_1}{\partial v}$ are continuous on the whole set Ω_1 .

3. Norm bound for the angular operator in a special case

Technically, this section is central in the paper. We aim at obtaining a norm bound for the angular operator X under condition $\sqrt{d(|\Delta| - d)} \leq \|V\| < \sqrt{d|\Delta|}$ which is outside of the scope of Theorem 2.6. In the proof we restrict ourselves, however, to the special case where the modulus $|X|$ of X has an eigenvalue coinciding with its norm $\||X|\| = \|X\|$.

In order to formulate the result we introduce another estimating function

$$M_2(D, d, v) := \sqrt{1 + \frac{2v^2}{D^2} - \frac{2}{D^2}\sqrt{dD - v^2}\sqrt{(D-d)D - v^2}}, \quad (D, d, v) \in \Omega_2, \quad (3.1)$$

where the set Ω_2 is defined by

$$\Omega_2 := \left\{ (D, d, v) \in \mathbb{R}^3 \mid D > 0, 0 < d \leq D/2, \sqrt{d(D-d)} \leq v < \sqrt{dD} \right\}.$$

Remark 3.1. Obviously, the function $M_2(D, d, v)$ is infinitely differentiable inside Ω_2 and continuous on Ω_2 . One verifies by inspection that

$$\min_{(D, d, v) \in \Omega_2} M_2(D, d, v) = 1, \quad \sup_{(D, d, v) \in \Omega_2} M_2(D, d, v) = \sqrt{2}, \quad (3.2)$$

and $M_2(D, d, v) = 1$ for any $(D, d, v) \in \partial\Omega_{12}$ where $\partial\Omega_{12}$ is the intersection (2.28) of the boundaries of Ω_1 and Ω_2 .

Theorem 3.2. *Assume Hypothesis 2.3. Assume in addition that*

$$\sqrt{d(|\Delta| - d)} \leq \|V\| < \sqrt{d|\Delta|}. \quad (3.3)$$

Let $X \in \mathcal{B}(\mathfrak{A}_0, \mathfrak{A}_1)$ be the unique solution to the Riccati equation (1.4) with the properties (2.22). If $|X|$ has an eigenvalue μ such that $\mu = \|X\|$, then the following bound holds:

$$\|X\| \leq M_2(|\Delta|, d, \|V\|), \quad (3.4)$$

where the function $M_2(D, d, v)$ is given by (3.1).

Proof. Throughout the proof we assume, without loss of generality, that the gap Δ is centered at zero, i.e. $\gamma_r = -\gamma_l = \gamma$; otherwise, one replaces A_0 and A_1 by $A'_0 = A_0 - cI$ and $A'_1 = A_1 - cI$, respectively, where $c = (\gamma_l + \gamma_r)/2$ is the center of Δ . The assumption that $\sigma_0 \subset \Delta = (-\gamma, \gamma)$ and $d = \text{dist}(\sigma_0, \sigma_1) > 0$ means that $\sigma_0 \subset [-a, a]$ with $a = \gamma - d$ and $\|A_0\| = a$.

Suppose that μ is an eigenvalue of $|X|$ such that $\mu = \|X\| = \||X|\|$ and let u , $\|u\| = 1$, be an eigenvector of $|X|$ associated with this eigenvalue, i.e. $|X|u = \mu u$. If $\mu = \|X\| \leq 1$ then, under condition (3.3), the bound (3.4) holds automatically by the first equality in (3.2). Further on in the proof we will always assume that $\mu > 1$.

Let Λ_0 be as in (2.10). Since $\text{spec}(\Lambda_0) = \text{spec}(A_0 + BX)$, from Theorem 2.4 it follows that $\text{spec}(\Lambda_0) = \omega_0$ and then (2.20) yields

$$0 \leq \|\Lambda_0 u\| \leq a + r_V < \gamma, \quad (3.5)$$

where r_V is given by (2.21) with $|\Delta| = 2\gamma = 2(a + d)$. At the same time

$$\|A_1 U u\|^2 + \|B U u\|^2 \geq \|A_1 U u\|^2 \geq \gamma^2,$$

taking into account that $u \in \text{Ran}(|X|)$, $\|u\| = 1$ and then $\|U u\| = 1$ by (2.6). Hence, by (3.5)

$$\|A_1 U u\|^2 + \|B U u\|^2 - \|\Lambda_0 u\|^2 \geq \gamma^2 - (a + r_V)^2 > 0$$

and the identity (2.9) in Lemma 2.2 implies

$$\mu^2 = \frac{\|A_0 u\|^2 + \|B^* u\|^2 - \|\Lambda_0 u\|^2}{\|A_1 U u\|^2 + \|B U u\|^2 - \|\Lambda_0 u\|^2}. \quad (3.6)$$

Since

$$\|A_0 u\| \leq a, \quad \|A_1 U u\| \geq \gamma, \quad \text{and} \quad \|B^* u\| \leq \|B\|,$$

from (3.6) it follows that

$$\mu^2 \leq \frac{a^2 + \|B\|^2 - \|\Lambda_0 u\|^2}{\gamma^2 + \|B U u\|^2 - \|\Lambda_0 u\|^2}. \quad (3.7)$$

Because of $\mu > 1$, from (3.7) one infers that

$$a^2 + \|B\|^2 > \gamma^2 + \|B U u\|^2. \quad (3.8)$$

As for the quantity $\|\Lambda_0 u\|$, in view of (3.5) we have two options: either

$$0 \leq \|\Lambda_0 u\| \leq a \quad (3.9)$$

or

$$a < \|\Lambda_0 u\| \leq a + r_V. \quad (3.10)$$

Since for any $s, t \in \mathbb{R}$ such that $t < s$ the function $f(x) := \frac{s-x}{t-x}$ is increasing at $x < t$, in the case (3.9) combining inequalities (3.7) and (3.8) yields

$$\mu^2 \leq \frac{\|B\|^2}{\gamma^2 + \|BUu\|^2 - a^2} \leq \frac{\|B\|^2}{\gamma^2 - a^2} \quad (\text{if } \|\Lambda_0 u\| \leq a). \quad (3.11)$$

In order to treat the case (3.10) properly, one notices that, due to (2.10),

$$\begin{aligned} \|\Lambda_0 u\| &= \|(I + |X|^2)^{1/2} (A_0 + BX)(I + |X|^2)^{-1/2} u\| \\ &\leq \frac{\sqrt{1 + \|X\|^2}}{\sqrt{1 + \mu^2}} \|A_0 u + \mu BUu\| \\ &= \|A_0 u\| + \mu \|BUu\|, \end{aligned} \quad (3.12)$$

taking into account that $|X|u = \mu u$ at the first step and that $\|X\| = \mu$ at the second. Since $\|A_0 u\| \leq a$, from (3.12) one deduces that, in the case (3.10), $\|BUu\| \geq \frac{1}{\mu} (\|\Lambda_0 u\| - a) > 0$ and then (3.7) implies

$$\mu^2 \leq \frac{a^2 + \|B\|^2 - \|\Lambda_0 u\|^2}{\gamma^2 + \frac{1}{\mu^2} (\|\Lambda_0 u\| - a)^2 - \|\Lambda_0 u\|^2} \quad (\text{if } \|\Lambda_0 u\| > a). \quad (3.13)$$

Inequality (3.13) transforms into

$$\mu^2 \leq \frac{\|B\|^2 + 2\|\Lambda_0 u\|(a - \|\Lambda_0 u\|)}{\gamma^2 - \|\Lambda_0 u\|^2} \quad (\text{if } \|\Lambda_0 u\| > a). \quad (3.14)$$

By combining (3.11) and (3.14) one arrives at the estimate

$$\mu^2 \leq \begin{cases} \varphi(a) & \text{if } \|\Lambda_0 u\| \leq a, \\ \varphi(\|\Lambda_0 u\|) & \text{if } \|\Lambda_0 u\| > a, \end{cases} \quad (3.15)$$

where the function $\varphi(z)$ for $z \in [0, \gamma]$ is defined by

$$\varphi(z) := \frac{\|B\|^2 + 2z(a - z)}{\gamma^2 - z^2}. \quad (3.16)$$

One observes that $\varphi(0) = \|B\|^2/\gamma^2 > 0$ and $\varphi(z) \rightarrow -\infty$ as $z \rightarrow \gamma - 0$ since

$$\|B\|^2 + 2\gamma(a - \gamma) = \|V\|^2 - d|\Delta| < 0$$

by hypothesis (3.3). Again taking into account (3.5), by (3.15) one concludes that in any case

$$\mu^2 \leq \max_{z \in [0, \gamma]} \varphi(z). \quad (3.17)$$

We notice that the function (3.16) already appeared in the proof of Lemma 2.1 in [3]. There is a single point z_0 within the interval $[0, \gamma]$ (in fact, $z_0 \in [0, a + r_V]$) where the derivative of this function is zero, namely

$$z_0 = \begin{cases} 0 & \text{if } a = 0, \\ \frac{2\gamma^2 - \|B\|^2}{2a} - \sqrt{\left(\frac{2\gamma^2 - \|B\|^2}{2a}\right)^2 - \gamma^2} & \text{if } a > 0. \end{cases} \quad (3.18)$$

At z_0 the function $\varphi(z)$ attains its maximum on $[0, \gamma]$, i.e.

$$\max_{z \in [0, \gamma]} \varphi(z) = \varphi(z_0). \quad (3.19)$$

By inspection, $\varphi(z_0) = M_2(2\gamma, \gamma - a, \|B\|)^2 = M_2(|\Delta|, d, \|V\|)^2$, where the function $M_2(D, d, v)$ is given by (3.1). Combining this with (3.17) and (3.19) completes the proof. \square

From the two estimating functions introduced in (2.25) and (3.1) we combine the total estimating function

$$M(D, d, v) := \begin{cases} M_1(D, d, v) & \text{if } 0 \leq v < \sqrt{d(D-d)}, \\ M_2(D, d, v) & \text{if } \sqrt{d(D-d)} \leq v < \sqrt{dD}, \end{cases} \quad (3.20)$$

which is considered on the union $\Omega := \Omega_1 \cup \Omega_2$ of the domains Ω_1 and Ω_2 ,

$$\Omega = \left\{ (D, d, v) \in \mathbb{R}^3 \mid D > 0, 0 < d \leq D/2, 0 \leq v < \sqrt{dD} \right\}.$$

Remark 3.3. By Remarks 2.7 and 3.1, the estimating function $M(D, d, v)$ is continuous and uniformly bounded on the whole set Ω . It also admits a continuous extension to the boundary $\partial\Omega$ of Ω (except for the intersection of $\partial\Omega$ with the D axis). It should be underlined, however, that the partial derivatives $\frac{\partial M(D, d, v)}{\partial D}$, $\frac{\partial M(D, d, v)}{\partial d}$, and $\frac{\partial M(D, d, v)}{\partial v}$ are discontinuous when, for $d < D/2$, the point (D, d, v) crosses the common boundary $\partial\Omega_{12}$ of the sets Ω_1 and Ω_2 .

4. Subspace variation bound in the general case.

Proof of Theorem 1

The norm bound for the angular operator X obtained in the previous section for the special case where $|X|$ has an eigenvalue equal to $\|X\|$ allows us to prove the following general subspace perturbation bound.

Theorem 4.1. *Assume Hypothesis 2.3. If $\|V\| < \sqrt{d|\Delta|}$, then*

$$\|E_A(\sigma_0) - E_L(\omega_0)\| \leq \sin(\arctan M(|\Delta|, d, \|V\|)), \quad (4.1)$$

where $\omega_0 = \text{spec}(L) \cap \Delta$ and $M(D, d, v)$ is the function defined by (3.20).

Proof. Assume, without loss of generality, that the gap Δ lies on the non-negative semiaxis, that is,

$$0 \leq \gamma_l < \gamma_r; \quad (4.2)$$

otherwise, one replaces A_0 and A_1 by $A'_0 = A_0 + (\frac{|\Delta|}{2} - c)I$ and $A'_1 = A_1 + (\frac{|\Delta|}{2} - c)I$, respectively, where $c = (\gamma_l + \gamma_r)/2$.

First, we consider the case where the spectral subspace \mathfrak{A}_0 is finite-dimensional. Theorem 2.4 (ii) ensures the existence of a unique angular operator X for the pair of subspaces $\mathfrak{A}_0 = \text{Ran}(E_A(\sigma_0))$ and $\mathfrak{L}_0 = \text{Ran}(E_L(\omega_0))$. Since $\dim(\mathfrak{A}_0) < \infty$, the operator X is of finite rank and so is its modulus $|X|$. Then there is an eigenvalue μ of $|X|$ such that $\mu = \||X|\| = \|X\|$. Hence, for $\sqrt{d(|\Delta| - d)} \leq \|V\| < \sqrt{d|\Delta|}$, the bound (4.1) follows by (1.5) and (3.20) from Theorem 3.2. For $\|V\| < \sqrt{d(|\Delta| - d)}$

this bound is implied by Theorem 2.6. Therefore, for the case where $\dim(\mathfrak{A}_0) < \infty$, the bound (4.1) has been proven.

If the subspace \mathfrak{A}_0 is infinite-dimensional, let $\{P_n^{(0)}\}_{n \in \mathbb{N}}$ be a sequence of finite-dimensional orthogonal projections in \mathfrak{A}_0 such that $\text{Ran}(P_n^{(0)}) \subset \mathfrak{A}_0$ and $s\text{-}\lim_{n \rightarrow \infty} P_n^{(0)} = I_{\mathfrak{A}_0}$. Using the projections $P_n^{(0)}$ we introduce the block diagonal operator matrices

$$A_n = \begin{pmatrix} P_n^{(0)} A_0 P_n^{(0)} & 0 \\ 0 & A_1 \end{pmatrix}, \quad \text{Dom}(A_n) := \text{Dom}(A) \quad (= \mathfrak{A}_0 \oplus \text{Dom}(A_1)),$$

which represent the corresponding truncations of the operator A with finite rank parts in \mathfrak{A}_0 . We also introduce the finite rank operators

$$V_n = \begin{pmatrix} 0 & P_n^{(0)} B \\ B^* P_n^{(0)} & 0 \end{pmatrix}$$

and set $L_n = A_n + V_n$, $\text{Dom}(L_n) := \text{Dom}(A_n) = \text{Dom}(A)$. The operators A_n and V_n are self-adjoint. Hence, so are the operators L_n .

Obviously, for any $\lambda \in \mathbb{C} \setminus \mathbb{R}$ the following operator identities hold:

$$(A_n - \lambda I)^{-1} - (A - \lambda I)^{-1} = (A_n - \lambda I)^{-1} S_n (A - \lambda I)^{-1}, \quad (4.3)$$

$$(L_n - \lambda I)^{-1} - (L - \lambda I)^{-1} = (L_n - \lambda I)^{-1} (S_n + V - V_n) (L - \lambda I)^{-1}, \quad (4.4)$$

where S_n is the bounded operator on \mathfrak{H} given by

$$S_n = \begin{pmatrix} A_0 - P_n^{(0)} A_0 P_n^{(0)} & 0 \\ 0 & 0 \end{pmatrix}. \quad (4.5)$$

By, e.g., [4, Theorem 2.5.2] we have $s\text{-}\lim_{n \rightarrow \infty} V_n = V$, $s\text{-}\lim_{n \rightarrow \infty} (P_n^{(0)} A_0 P_n^{(0)}) = A_0$, and then, due to (4.3)–(4.5),

$$s\text{-}\lim_{n \rightarrow \infty} (A_n - \lambda I)^{-1} = (A - \lambda I)^{-1} \quad \text{and} \quad s\text{-}\lim_{n \rightarrow \infty} (L_n - \lambda I)^{-1} = (L - \lambda I)^{-1} \quad (4.6)$$

for any $\lambda \in \mathbb{C} \setminus \mathbb{R}$, which means that both sequences $\{A_n\}_{n \in \mathbb{N}}$ and $\{L_n\}_{n \in \mathbb{N}}$ are convergent in strong resolvent sense (see, e.g., [14, Section VIII.7]).

Let \widehat{A}_n and \widehat{V}_n denote the parts of the operators A_n and V_n associated with their reducing subspace

$$\widehat{\mathfrak{H}}_n = \widehat{\mathfrak{A}}_0^{(n)} \oplus \mathfrak{A}_1, \quad (4.7)$$

where $\widehat{\mathfrak{A}}_0^{(n)} = \text{Ran}(P_n^{(0)})$. Clearly, the operator \widehat{A}_n is block diagonal with respect to the decomposition (4.7), $\text{Dom}(\widehat{A}_n) = \widehat{\mathfrak{A}}_0^{(n)} \oplus \text{Dom}(A_1)$, and $\widehat{A}_n|_{\mathfrak{A}_1} = A_1$. Further, for the spectral set $\widehat{\sigma}_0^{(n)} := \text{spec}(\widehat{A}_n|_{\widehat{\mathfrak{A}}_0^{(n)}})$ we have the inclusion

$$\widehat{\sigma}_0^{(n)} \subset [\gamma_l + d, \gamma_r - d] \quad (4.8)$$

and, thus,

$$d_n := \text{dist}\left(\text{spec}(\widehat{A}_n|_{\widehat{\mathfrak{A}}_0^{(n)}}), \text{spec}(\widehat{A}_n|_{\mathfrak{A}_1})\right) = \text{dist}(\widehat{\sigma}_0^{(n)}, \sigma_1) \geq d. \quad (4.9)$$

By its construction, the finite rank operator \widehat{V}_n is off-diagonal with respect to the decomposition (4.7) and

$$\|\widehat{V}_n\| \leq \|V\|. \quad (4.10)$$

By the hypothesis we have $\|V\| < \sqrt{d|\Delta|}$. Hence, from (4.10) and (4.9) it follows that $\|\widehat{V}_n\| < \sqrt{d|\Delta|} \leq \sqrt{d_n|\Delta|}$. Then Theorem 2.4 (i) implies that the spectrum of $\widehat{L}_n := \widehat{A}_n + \widehat{V}_n$ consists of two disjoint components $\widehat{\omega}_0^{(n)}$ and $\widehat{\omega}_1^{(n)}$ such that

$$\widehat{\omega}_0^{(n)} \subset [\gamma_l + d_n - r_V^{(n)}, \gamma_r - d_n + r_V^{(n)}] \subset \Delta \quad \text{and} \quad \widehat{\omega}_1^{(n)} \subset \mathbb{R} \setminus \Delta, \quad (4.11)$$

where $r_V^{(n)}$ is given by

$$r_V^{(n)} = \|\widehat{V}_n\| \tan \left(\frac{1}{2} \arctan \frac{2\|\widehat{V}_n\|}{|\Delta| - d_n} \right).$$

Since $d \leq d_n \leq \frac{|\Delta|}{2}$ and \widehat{V}_n satisfies (4.10), one easily verifies that $d_n - r_V^{(n)} \geq d - r_V$ with r_V given by (2.21). Therefore, from the first inclusion in (4.11) it follows that

$$\widehat{\omega}_0^{(n)} \subset [\gamma_l + d - r_V, \gamma_r - d + r_V] \quad \text{for any } n \in \mathbb{N}. \quad (4.12)$$

Furthermore, since the spectral subspace $\widehat{\mathfrak{A}}_0^{(n)} = \text{Ran}(\widehat{E}_{\widehat{A}_n}(\widehat{\sigma}_0^{(n)}))$ is finite-dimensional, the bound (4.1) applies to the spectral projections $\widehat{E}_{\widehat{A}_n}(\widehat{\sigma}_0^{(n)})$ and $\widehat{E}_{\widehat{L}_n}(\widehat{\omega}_0^{(n)})$:

$$\|\widehat{E}_{\widehat{A}_n}(\widehat{\sigma}_0^{(n)}) - \widehat{E}_{\widehat{L}_n}(\widehat{\omega}_0^{(n)})\| \leq \sin(\arctan M(|\Delta|, d_n, \|\widehat{V}_n\|)). \quad (4.13)$$

Observing that the function $M(D, d, v)$ is monotonously increasing as the second argument decreases and/or the third one increases, by (4.9) and (4.10) from (4.13) one infers that

$$\|\widehat{E}_{\widehat{A}_n}(\widehat{\sigma}_0^{(n)}) - \widehat{E}_{\widehat{L}_n}(\widehat{\omega}_0^{(n)})\| \leq \sin(\arctan M(|\Delta|, d, \|V\|)). \quad (4.14)$$

Now for an arbitrary ε such that $0 < \varepsilon < d - r_V$ we set $\Sigma_\varepsilon := (\gamma_l + \varepsilon, \gamma_r - \varepsilon)$. Obviously, by (4.8) and (4.12), the open interval Σ_ε contains both sets $\widehat{\sigma}_0^{(n)}$ and $\widehat{\omega}_0^{(n)}$. Hence, $\widehat{E}_{\widehat{A}_n}(\widehat{\sigma}_0^{(n)}) = \widehat{E}_{\widehat{A}_n}(\Sigma_\varepsilon)$ and $\widehat{E}_{\widehat{L}_n}(\widehat{\omega}_0^{(n)}) = \widehat{E}_{\widehat{L}_n}(\Sigma_\varepsilon)$. Then inequality (4.14) may be rewritten as

$$\|\widehat{E}_{\widehat{A}_n}(\Sigma_\varepsilon) - \widehat{E}_{\widehat{L}_n}(\Sigma_\varepsilon)\| \leq \sin(\arctan M(|\Delta|, d, \|V\|)). \quad (4.15)$$

Clearly, the spectrum of the part $L_n|_{\widehat{\mathfrak{H}}_n^\perp}$ of the operator L_n associated with its reducing subspace $\widehat{\mathfrak{H}}_n^\perp = \mathfrak{H} \ominus \widehat{\mathfrak{H}}_n$ consists of the single point zero and the same holds for the spectrum of the restriction $A_n|_{\widehat{\mathfrak{H}}_n^\perp}$, i.e.

$$\text{spec}(L_n|_{\widehat{\mathfrak{H}}_n^\perp}) = \text{spec}(A_n|_{\widehat{\mathfrak{H}}_n^\perp}) = \{0\}. \quad (4.16)$$

By (4.2) this means that none of the sets $\text{spec}(L_n|_{\widehat{\mathfrak{H}}_n^\perp})$ and $\text{spec}(A_n|_{\widehat{\mathfrak{H}}_n^\perp})$ intersects the interval Σ_ε . Hence, (4.15) yields

$$\|\widehat{E}_{A_n}(\Sigma_\varepsilon) - \widehat{E}_{L_n}(\Sigma_\varepsilon)\| \leq \sin(\arctan M(|\Delta|, d, \|V\|)). \quad (4.17)$$

Meanwhile, equalities (4.16) considered together with the inclusions (4.8) and (4.12) imply

$$\begin{aligned} (\gamma_l, \gamma_l + d) &\subset \rho(A_n) \text{ and } (\gamma_r - d, \gamma_r) \subset \rho(A_n), \\ (\gamma_l, \gamma_l + d - r_V) &\subset \rho(L_n) \text{ and } (\gamma_r - d + r_V, \gamma_r) \subset \rho(L_n). \end{aligned}$$

Then, from the strong resolvent convergence (4.6) of the sequences $\{A_n\}_{n \in \mathbb{N}}$ and $\{L_n\}_{n \in \mathbb{N}}$, it follows (see, e.g., [14, Theorem VIII.24]) that for any ε such that $0 < \varepsilon < d - r_V$

$$s\text{-}\lim_{n \rightarrow \infty} E_{A_n}(\Sigma_\varepsilon) = E_A(\Sigma_\varepsilon) \text{ and } s\text{-}\lim_{n \rightarrow \infty} E_{L_n}(\Sigma_\varepsilon) = E_L(\Sigma_\varepsilon).$$

Passing to the limit as $n \rightarrow \infty$ in (4.17), one obtains

$$\|E_A(\Sigma_\varepsilon) - E_L(\Sigma_\varepsilon)\| \leq \sin(\arctan M(|\Delta|, d, \|V\|)),$$

which is equivalent to (4.1) since both spectral sets σ_0 and ω_0 are subsets of the interval Σ_ε (see Theorem 2.4 (i)). \square

Remark 4.2. The bound (4.1) is sharp. For $\|V\| < \sqrt{d(|\Delta| - d)}$, this has been established in [13] (see [13, Remark 5.6 (i)]). For $\sqrt{d(|\Delta| - d)} \leq \|V\| < \sqrt{d|\Delta|}$ the sharpness of (4.1) is proven by [3, Remark 2.3]. For convenience of the reader, below we reproduce the corresponding examples from [3] and [13] that prove the optimality of the bound (4.1).

Example 4.3 ([3]). Let $\mathfrak{H}_0 = \mathbb{C}$ and $\mathfrak{H}_1 = \mathbb{C}^2$. Assuming that $0 \leq a < \gamma$ and $b_1, b_2 \geq 0$, we set

$$A_0 = a, \quad A_1 = \begin{pmatrix} -\gamma & 0 \\ 0 & \gamma \end{pmatrix}, \text{ and } B = (b_1 \ b_2).$$

The operators (3×3 matrices) A , V , and L are defined on $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_1 = \mathbb{C}^3$ by equalities (2.1), (2.2), and (2.3), respectively. The spectrum $\sigma_0 = \{a\}$ of A_0 lies in the gap $\Delta = (-\gamma, \gamma)$ of the spectrum $\sigma_1 = \{-\gamma, \gamma\}$ of A_1 . Also notice that $d = \text{dist}(\sigma_0, \sigma_1) = \gamma - a$ and $|\Delta| = 2\gamma$.

First, consider the case where $b_1 = 0$ and, thus, $\|V\| = \|B\| = b_2$. In this case, for any $b_2 \geq 0$ satisfying $b_2^2 < 2\gamma(\gamma + a)$, i.e., for $\|V\| < \sqrt{|\Delta|(|\Delta| - d)}$ the matrix L has a single eigenvalue within the interval Δ ; two other eigenvalues of L are in $\mathbb{R} \setminus \Delta$. For the difference of the eigenprojections $E_A(\sigma_0)$ and $E_L(\omega_0)$ we have

$$\|E_A(\sigma_0) - E_L(\omega_0)\| = \frac{2\|V\|}{d + \sqrt{d^2 + 4\|V\|^2}}, \quad (4.18)$$

where, as usually, $\omega_0 = \text{spec}(L) \cap \Delta$. Since $\sqrt{|\Delta|(|\Delta| - d)} > \frac{1}{2}\sqrt{d(|\Delta| - 2d)}$, equality (4.18) proves the optimality of the bound (4.1) for

$$\|V\| \leq \frac{1}{2}\sqrt{d(|\Delta| - 2d)}$$

(see definition (2.26) of the restriction $M|_{\Omega_1^{(0)}} = M_1|_{\Omega_1^{(0)}}$ of the function M onto the set $\Omega_1^{(0)}$).

Second, assume that b is a positive number such that

$$\sqrt{\gamma^2 - a^2} \leq b < \sqrt{2\gamma(\gamma - a)} \quad (4.19)$$

and let (cf. formula (3.18))

$$z_0 := \begin{cases} 0 & \text{if } a = 0, \\ \frac{2\gamma^2 - b^2}{2a} - \sqrt{\left(\frac{2\gamma^2 - b^2}{2a}\right)^2 - \gamma^2} & \text{if } a > 0. \end{cases} \quad (4.20)$$

The second inequality in (4.19) implies $z_0 \in \Delta$. Note that, under condition (4.19), for

$$t := \frac{1}{2\gamma b^2} (b^2(\gamma - z_0) + (\gamma^2 - z_0^2)(a - z_0))$$

one has $0 \leq t < 1$ (actually, $t \leq 1/2$). Then set

$$b_1 := \sqrt{1-t}b, \quad b_2 := \sqrt{t}b. \quad (4.21)$$

If b_1 and b_2 are introduced by (4.21), then $\|V\| = b$. Furthermore, in this case (4.19) is equivalent to

$$\sqrt{d(|\Delta| - d)} \leq \|V\| < \sqrt{d|\Delta|} \quad (4.22)$$

and the number z_0 given by (4.20) represents the single eigenvalue of the matrix L within the interval Δ , that is, $\omega_0 = \text{spec}(L) \cap \Delta = \{z_0\}$. An explicit computation of the only eigenvector of L corresponding to the eigenvalue z_0 results in

$$\|E_A(\sigma_0) - E_L(\omega_0)\| = \sin(\arctan M_2(|\Delta|, d, \|V\|)). \quad (4.23)$$

Taking into account definition (3.20) of the function M , equality (4.23) proves the sharpness of the bound (4.1) for V satisfying (4.22).

Example 4.4 ([13]). Let $\mathfrak{H}_0 = \mathfrak{H}_1 = \mathbb{C}^2$. Assume that A_0, A_1 , and B are 2×2 matrices given respectively by

$$A_0 = \begin{pmatrix} -a & 0 \\ 0 & a \end{pmatrix}, \quad A_1 = \begin{pmatrix} -\gamma & 0 \\ 0 & \gamma \end{pmatrix}, \quad \text{and } B = \begin{pmatrix} b_1 & b_2 \\ b_2 & b_1 \end{pmatrix},$$

where $0 \leq a < \gamma$ and $b_1, b_2 \geq 0$. Let the operators (4×4 matrices) A , V , and L be defined on $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_1 = \mathbb{C}^4$ by equalities (2.1), (2.2), and (2.3), respectively. Clearly, the spectrum $\sigma_0 = \{-a, a\}$ of A_0 lies in the gap $\Delta = (-\gamma, \gamma)$ of the spectrum $\sigma_1 = \{-\gamma, \gamma\}$ of A_1 . We also note that $d = \text{dist}(\sigma_0, \sigma_1) = \gamma - a$, $|\Delta| = 2\gamma$, and $\|V\| = \|B\| = b_1 + b_2$.

One verifies by inspection that the 2×2 matrix

$$X = \begin{pmatrix} \varkappa_1 & \varkappa_2 \\ -\varkappa_2 & -\varkappa_1 \end{pmatrix},$$

where

$$\begin{aligned} \varkappa_1 &= \frac{2b_1\sqrt{(\gamma+a)^2 + 4b_2^2}}{(\gamma+a)\sqrt{(\gamma-a)^2 + 4b_1^2} + (\gamma-a)\sqrt{(\gamma+a)^2 + 4b_2^2}}, \\ \varkappa_2 &= \frac{2b_2\sqrt{(\gamma-a)^2 + 4b_1^2}}{(\gamma+a)\sqrt{(\gamma-a)^2 + 4b_1^2} + (\gamma-a)\sqrt{(\gamma+a)^2 + 4b_2^2}}, \end{aligned}$$

is a solution to the operator Riccati equation (1.4). Moreover, under condition $\|V\| < \sqrt{d|\Delta|} = \sqrt{2\gamma(\gamma-a)}$ the spectrum of $A_0 + BX$ lies in the interval Δ , while both

eigenvalues of $A_1 - B^*X^*$ are in $\mathbb{R} \setminus \Delta$. Applying, e.g., [9, Theorem 2.3] one concludes that the graph subspace $\mathcal{G}(X) = \{x \oplus Xx \mid x \in \mathfrak{H}_0\}$ is the spectral subspace of L associated with the spectral set $\omega_0 = \text{spec}(L) \cap \Delta$. By (1.5) this yields

$$\|E_A(\sigma_0) - E_L(\omega_0)\| = \sin(\arctan(\varkappa_1 + \varkappa_2)), \quad (4.24)$$

taking into account that $\|X\| = \varkappa_1 + \varkappa_2$.

Now pick up an arbitrary b satisfying

$$\frac{1}{2}\sqrt{2(\gamma-a)a} < b < \sqrt{\gamma^2 - a^2} \quad (4.25)$$

and set

$$b_1 = \frac{1}{2}(b + \beta) \text{ and } b_2 = \frac{1}{2}(b - \beta), \quad (4.26)$$

where

$$\beta = \begin{cases} 0 & \text{if } a = 0, \\ \frac{1}{a} \left(\sqrt{\gamma^2 b^2 + a^2(\gamma^2 - a^2 - b^2)} - \gamma b \right) & \text{if } a > 0. \end{cases}$$

The positivity of b_1 is obvious. The positivity of b_2 is implied by the first inequality in (4.25). Since $\|V\| = \|B\| = b_1 + b_2 = b$, the pair of inequalities (4.25) is equivalent to

$$\frac{1}{2}\sqrt{d(|\Delta| - 2d)} < \|V\| < \sqrt{d(|\Delta| - d)}. \quad (4.27)$$

Evaluation of the sum $\varkappa_1 + \varkappa_2$ for b_1 and b_2 given by (4.26) reduces (4.24) to

$$\|E_A(\sigma_0) - E_L(\omega_0)\| = \sin(\arctan M_{\Omega_1^{(1)}}(|\Delta|, d, \|V\|)), \quad (4.28)$$

where $M_{\Omega_1^{(1)}} := M_1|_{\Omega_1^{(1)}} = M|_{\Omega_1^{(1)}}$ is the restriction (2.27) of the function M onto the set $\Omega_1^{(1)}$. Therefore, equality (4.28) proves the sharpness of the bound (4.1) for V satisfying (4.27).

Theorem 1 is nothing but a corollary of Theorem 4.1.

Proof of Theorem 1. Set $\mathfrak{A}_0 = \text{Ran}(E_A(\sigma_0))$ and $\mathfrak{A}_1 = \text{Ran}(E_A(\sigma_1))$. With respect to the orthogonal decomposition $\mathfrak{H} = \mathfrak{A}_0 \oplus \mathfrak{A}_1$ the operators A and V are block operator matrices of the form (2.1) and (2.2), respectively. The length of the gap Δ satisfies the estimate $|\Delta| \geq 2d$ and, hence, condition (1.2) implies $\|V\| < \sqrt{d|\Delta|}$. Then by Theorem 4.1 we have estimate (4.1). It remains to observe that, given the values of $\|V\|$ and d satisfying (1.2), $M(D, d, \|V\|)$ is a non-increasing function of the variable D , $D \geq 2d$. For D varying in the interval $[2d, \infty)$ it attains its maximal value at $D = 2d$ and this value equals

$$\max_{D: D \geq 2d} M(D, d, \|V\|) = M(2d, d, \|V\|) = \frac{\|V\|}{d}.$$

Hence, (4.1) yields (1.3), completing the proof. \square

Remark 4.5. Example 2.4 in [3] (representing a version of Example 4.3 for $a = 0$ and $b_1 = b_2 = b/\sqrt{2}$, where $b \geq 0$) shows that the bound (1.3) is sharp.

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